

# Finding Almost Squares III

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## Abstract

An almost square of type 2 is an integer  $n$  that can be factored in two different ways as  $n = a_1b_1 = a_2b_2$  with  $a_1, a_2, b_1, b_2 \approx \sqrt{n}$ . In this paper, we shall improve upon previous result on short intervals containing an almost square of type 2. This leads to an inquiry of finding a short interval around  $x$  that contains an integer divisible by some integer in  $[x^c, 2x^c]$  with  $0 < c < 1$ .

## 1 Introduction and main results

In [1] and [2], the author started studying almost square, an integer  $n$  that can be factored as  $n = ab$  with  $a, b$  close to  $\sqrt{n}$ . More specifically, for  $0 \leq \theta \leq 1/2$  and  $C > 0$ ,

**Definition 1** *An integer  $n$  is a  $(\theta, C)$ -almost square of type 1 if  $n = ab$  for some integers  $a, b$  in the interval  $[n^{1/2} - Cn^\theta, n^{1/2} + Cn^\theta]$ .*

**Definition 2** *An integer  $n$  is a  $(\theta, C)$ -almost square of type 2 if  $n = a_1b_1 = a_2b_2$  for some integers  $a_1 < a_2 \leq b_2 < b_1$  in the interval  $[n^{1/2} - Cn^\theta, n^{1/2} + Cn^\theta]$ .*

Let  $x$  be a large positive real number. Following [1] and [2], we are interested in finding almost squares of type 1 or 2 near to  $x$ . In particular, given  $0 \leq \theta \leq 1/2$ , we want to find “admissible”  $\phi_i \geq 0$  (as small as possible) such that, for some constants  $C_{\theta,i}, D_{\theta,i} > 0$ , the interval  $[x - D_{\theta,i}x^{\phi_i}, x + D_{\theta,i}x^{\phi_i}]$  contains a  $(\theta, C_{\theta,i})$ -almost square of type  $i$  ( $i = 1, 2$ ) for all large  $x$ .

### Definition 3

$$f(\theta) := \inf \phi_1 \quad \text{and} \quad g(\theta) := \inf \phi_2$$

where the infima are taken over all the “admissible”  $\phi_i$  ( $i = 1, 2$ ) respectively.

Clearly  $f$  and  $g$  are non-increasing functions of  $\theta$ . Summarizing the results in [1] and [2], we have

$$f(\theta) \begin{cases} = 1/2, & \text{if } 0 \leq \theta < 1/4, \\ = 1/4, & \text{if } \theta = 1/4, \\ = 1/2 - \theta, & \text{if } 1/4 \leq \theta \leq 3/10 \text{ and a conjectural upper bound on} \\ & \text{certain average of twisted incomplete Salie sum is true,} \\ \geq 1/2 - \theta, & \text{if } 1/4 \leq \theta \leq 1/2. \end{cases}$$
$$g(\theta) \begin{cases} \text{does not exist,} & \text{if } 0 \leq \theta < 1/4, \\ \geq 1 - 2\theta, & \text{if } 1/4 \leq \theta \leq 1/2, \\ \leq 1 - \theta, & \text{if } 1/4 \leq \theta \leq 1/3. \end{cases}$$
$$f(\theta) = \begin{cases} 1/2, & \text{if } 0 \leq \theta < 1/4, \\ 1/2 - \theta, & \text{if } 1/4 \leq \theta \leq 1/2; \end{cases}$$
$$g(\theta) = \begin{cases} \text{does not exist,} & \text{if } 0 \leq \theta < 1/4, \\ 1 - 2\theta, & \text{if } 1/4 \leq \theta \leq 1/2. \end{cases}$$
$$\begin{array}{ll} (i) & g(1/4) \leq 5/8, \\ (ii) & g(\theta) \leq 9/16, \quad \text{if } 5/16 \leq \theta \leq 1/2, \\ (iii) & g(\theta) \leq 17/32, \quad \text{if } 5/16 \leq \theta \leq 1/2, \\ (iv) & g(\theta) \leq 1/2, \quad \text{if } 1/3 < \theta \leq 1/2, \\ (v) & g(\theta) \leq 1/2, \quad \text{if } 743/2306 < \theta \leq 1/2. \end{array}$$

The plot shows a function  $g(\theta)$  defined on the interval  $\theta \in [0, 1/2]$ . The function is piecewise linear, consisting of several horizontal segments and downward-sloping segments. The horizontal axis is marked with  $\theta = 0, 1/4, 5/16, 743/2306, 1/2$ . The vertical axis is marked with  $g(\theta) = 0, 1/2, 5/8, 1$ . Arrows indicate the direction of the function's path between segments.

The above picture summarizes Theorems 2 and 3. The thin line segments are the upper and lower bounds from Theorem 2. The thick line segments are the upper bounds from Theorem 3. The next challenge is to beat the  $\frac{1}{2}$  upper bound for  $g(\theta)$ .

**Some Notations:** Throughout the paper,  $\epsilon$  denotes a small positive number. Both  $f(x) = O(g(x))$  and  $f(x) \ll g(x)$  mean that  $|f(x)| \leq Cg(x)$  for some constant  $C > 0$ . Moreover  $f(x) = O_\lambda(g(x))$  and  $f(x) \ll_\lambda g(x)$  mean that the implicit constant  $C = C_\lambda$  may depend on the parameter  $\lambda$ . Finally  $f(x) \asymp g(x)$  means that  $f(x) \ll g(x)$  and  $g(x) \ll f(x)$ .

## 2 Proof of Theorem 3 (i)

Let  $1/4 \leq \theta \leq 1/2$ . From [2], we recall that a  $(\theta, C)$ -almost square of type 2 must be of the form

$$n = (d_1 e_1)(d_2 e_2) = (d_1 e_2)(d_2 e_1)$$

where  $a_1 = d_1 e_1$ ,  $b_1 = d_2 e_2$ ,  $a_2 = d_1 e_2$ ,  $b_2 = d_2 e_1$ ;  $n^{1/2} - Cn^\theta \leq a_1 < a_2 \leq b_2 < b_1 \leq n^{1/2} + Cn^\theta$ ;

$$\frac{1}{2C}n^{\frac{1}{2}-\theta} - \frac{1}{2} \leq d_1, d_2, e_1, e_2 \leq 2Cn^\theta, \quad e_2 - e_1 \leq 2C\frac{n^\theta}{d_2}, \quad d_2 - d_1 \leq 2C\frac{n^\theta}{e_2}.$$

Let  $1 \leq k \ll 1$  be any integer. By  $\theta = 1/4$  case in Theorem 1, for some constant  $C > 0$ , we can find integers  $d, e \in [x^{1/4} - Cx^{1/8}, x^{1/4} + Cx^{1/8}]$  such that

$$de = x^{1/2} - 2kx^{1/4} + O(x^{1/8}).$$

Then

$$(d + 2k)(e + 2k) = de + 2k(d + e) + k^2 = x^{1/2} + 2kx^{1/4} + O(x^{1/8}),$$

and

$$de(d + 2k)(e + 2k) = x - 4k^2x^{1/2} + O(x^{5/8}) = x + O(x^{5/8}).$$

This gives  $g(1/4) \leq 5/8$ .

## 3 Proof of Theorem 3 (ii)

The key idea is the identity

$$ab = \left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2$$

as used in [1]. Using this identity,

$$\begin{aligned}
d_1 e_1 d_2 e_2 &= \left[ \left( \frac{d_2 + d_1}{2} \right)^2 - \left( \frac{d_2 - d_1}{2} \right)^2 \right] \left[ \left( \frac{e_2 + e_1}{2} \right)^2 - \left( \frac{e_2 - e_1}{2} \right)^2 \right] \\
&= \left( \frac{d_2 + d_1}{2} \right)^2 \left( \frac{e_2 + e_1}{2} \right)^2 - \left( \frac{d_2 - d_1}{2} \right)^2 \left( \frac{e_2 + e_1}{2} \right)^2 \\
&\quad - \left( \frac{e_2 - e_1}{2} \right)^2 \left( \frac{d_2 + d_1}{2} \right)^2 + \left( \frac{d_2 - d_1}{2} \right)^2 \left( \frac{e_2 - e_1}{2} \right)^2 \\
&=: G^2 H^2 - g^2 H^2 - h^2 G^2 + g^2 h^2
\end{aligned}$$

where  $G = \frac{d_2 + d_1}{2}$ ,  $H = \frac{e_2 + e_1}{2}$ ,  $g = \frac{d_2 - d_1}{2}$  and  $h = \frac{e_2 - e_1}{2}$ . Now we want

$$\begin{aligned}
x &\approx d_1 e_1 d_2 e_2 = G^2 H^2 - g^2 H^2 - h^2 G^2 + g^2 h^2 \\
G^2 H^2 - x &\approx g^2 H^2 + h^2 G^2 - g^2 h^2 \\
(GH - \sqrt{x})(GH + \sqrt{x}) &\approx g^2 H^2 + h^2 G^2 - g^2 h^2 \tag{1}
\end{aligned}$$

By  $\theta = 1/4$  case in Theorem 1, for some constant  $C > 0$ , there exist integers  $G, H \in [x^{1/4} - Cx^{1/16}, x^{1/4} + Cx^{1/16}]$  such that  $0 < GH - \sqrt{x} \asymp x^{1/8}$ . Then the left hand side of (1) is  $\asymp x^{1/2+1/8}$ . As for the right hand side of (1), observe that, for fixed  $h$  (say  $h = 1$ ), the increment

$$[(i+1)^2 H^2 + h^2 G^2 - (i+1)^2 h^2] - [i^2 H^2 + h^2 G^2 - i^2 h^2] = (2i+1)H^2 - (2i+1)h^2 \asymp x^{1/2} i.$$

Now observe that

$$\begin{aligned}
&g^2 H^2 + h^2 G^2 - g^2 h^2 \\
&= h^2 G^2 + \sum_{0 \leq i < g} [(i+1)^2 H^2 + h^2 G^2 - (i+1)^2 h^2] - [i^2 H^2 + h^2 G^2 - i^2 h^2] \\
&\asymp x^{1/2} \sum_{1 \leq i < g} i \asymp g^2 x^{1/2}.
\end{aligned}$$

Therefore, for some integer  $1 \leq g \asymp x^{1/16}$ ,

$$|\text{Right hand side of (1)} - \text{Left hand side of (1)}| \ll x^{1/2} g \ll x^{1/2+1/16}.$$

This gives

$$|x - (G^2 - g^2)(H^2 - h^2)| \ll x^{1/2+1/16}$$

or

$$|x - d_1 d_2 e_1 e_2| = |x - (G - g)(G + g)(H - h)(H + h)| \ll x^{1/2+1/16}.$$

Consequently, with

$$\begin{aligned}
a_1 &= d_1 e_1 = (G - g)(H - h) = x^{1/2} + O(x^{1/4+1/16}), \\
b_1 &= d_2 e_2 = (G + g)(H + h) = x^{1/2} + O(x^{1/4+1/16}), \\
a_2 &= d_1 e_2 = (G - g)(H + h) = x^{1/2} + O(x^{1/4+1/16}), \\
b_2 &= d_2 e_1 = (G + g)(H - h) = x^{1/2} + O(x^{1/4+1/16}),
\end{aligned}$$

we have a  $(\theta, C')$ -almost square  $n = a_1b_1 = a_2b_2$  of type 2 in the interval  $[x - C''x^{1/2+1/16}, x + C''x^{1/2+1/16}]$  for some  $C', C'' > 0$ . This proves that  $g(\theta) \leq 9/16$  for  $\theta \geq 1/4 + 1/16 = 5/16$ .

## 4 Proof of Theorem 3 (iii)

This time we try to approximate the left hand side of (1) by the quadratic form  $g^2H^2 + h^2G^2$  directly. As in the proof of Theorem 3 (iii), for some  $C > 0$ , there exist integers  $x^{1/4} - Cx^{1/16} \leq G, H \leq x^{1/4} + Cx^{1/16}$  such that  $0 < GH - \sqrt{x} \asymp x^{1/8}$ . The left hand side of (1) is  $\asymp x^{1/2+1/8}$ . Without loss of generality,  $G \leq H$ . Then  $g^2H^2 + h^2G^2 = G^2(g^2 + h^2) + (H^2 - G^2)g^2$ . Observe that  $0 \leq H^2 - G^2 = (H - G)(H + G) \ll x^{1/4+1/16}$ . By elementary argument, for any real number  $X > 0$ , we can find a sum of two squares  $g^2 + h^2$  such that  $|X - (g^2 + h^2)| \ll X^{1/4}$ . In particular, we can find  $1 \leq g, h \ll x^{1/16}$  such that

$$\left| \frac{(GH - \sqrt{x})(GH + \sqrt{x})}{G^2} - (g^2 + h^2) \right| \ll x^{1/32}.$$

This implies

$$\begin{aligned} & |(GH - \sqrt{x})(GH + \sqrt{x}) - (g^2H^2 + h^2G^2 - g^2h^2)| \\ & \leq |(GH - \sqrt{x})(GH + \sqrt{x}) - G^2(g^2 + h^2)| + |(H^2 - G^2)g^2| + |g^2h^2| \ll x^{1/2+1/32}. \end{aligned}$$

Hence

$$|x - d_1d_2e_1e_2| = |x - (G - g)(G + g)(H - h)(H + h)| \ll x^{1/2+1/32}.$$

Consequently, with

$$\begin{aligned} a_1 &= d_1e_1 = (G - g)(H - h) = x^{1/2} + O(x^{1/4+1/16}), \\ b_1 &= d_2e_2 = (G + g)(H + h) = x^{1/2} + O(x^{1/4+1/16}), \\ a_2 &= d_1e_2 = (G - g)(H + h) = x^{1/2} + O(x^{1/4+1/16}), \\ b_2 &= d_2e_1 = (G + g)(H - h) = x^{1/2} + O(x^{1/4+1/16}), \end{aligned}$$

there is a  $(\theta, C')$ -almost square  $n = a_1b_1 = a_2b_2$  of type 2 in the interval  $[x - C''x^{1/2+1/32}, x + C''x^{1/2+1/32}]$  for some  $C', C'' > 0$ . This proves that  $g(\theta) \leq 17/32$  for  $\theta \geq 1/4 + 1/16 = 5/16$ .

## 5 Proof of Theorem 3 (iv)

Let  $1/2 \leq \phi \leq 1$ . Observe that, for large  $x$ , the interval  $[x + x^{1-\phi}, x + 2x^{1-\phi}]$  contains an integer  $n$  which is divisible by an integer  $a \in [x^{1-\phi}/2, x^{1-\phi}]$ . In particular  $n = ab$  with integer  $b \in [x^\phi, 3x^\phi]$ .

Again we use (1). Instead of having  $G, H$  close to  $x^{1/4}$  in the proof of Theorem 3 (iii), we want

$$G \approx x^{(1-\phi)/2} \text{ and } H \approx x^{\phi/2} \text{ for some } 1/2 < \phi < 2/3.$$

By the observation at the beginning of this section, we can find  $H \in [x^{\phi/2}, 3x^{\phi/2}]$  and  $G \in [x^{(1-\phi)/2}/2, x^{(1-\phi)/2}]$  such that  $0 < GH - \sqrt{x} \asymp x^{(1-\phi)/2}$ . Then the left hand side of (1),  $L = (GH - \sqrt{x})(GH + \sqrt{x}) \asymp x^{1-\phi/2}$ .

Firstly we approximate  $L$  by  $g^2H^2$ . For some choice of  $g \asymp x^{1/2-3\phi/4}$ , we have  $0 < L - g^2H^2 \asymp gH^2 \asymp x^{1/2+\phi/4}$ . Note that  $1/2 - 3\phi/4 > 0$  as  $\phi < 2/3$ .

Secondly we approximate  $L - g^2H^2$  by  $h^2G^2$ . For some choice of  $h \asymp x^{5\phi/8-1/4}$ , we have  $|L - g^2H^2 - h^2G^2| \ll hG^2 \asymp x^{3/4-3\phi/8}$ . Note that  $5\phi/8 - 1/4 > 0$  as  $\phi > 1/2$ .

Thirdly, observe that  $g^2h^2 \ll x^{1/2-\phi/4} \ll x^{3/4-3\phi/8}$  as  $\phi < 2$ . Therefore,  $|L - g^2H^2 - h^2G^2 + g^2h^2| \ll x^{3/4-3\phi/8}$  which gives

$$|x - d_1d_2e_1e_2| = |x - (G - g)(G + g)(H - h)(H + h)| \ll x^{3/4-3\phi/8}.$$

Consequently, as  $1/2 < \phi < 2/3$ , with

$$\begin{aligned} a_1 &= d_1e_1 = (G - g)(H - h) = x^{1/2} + O(x^{1/2-\phi/4}), \\ b_1 &= d_2e_2 = (G + g)(H + h) = x^{1/2} + O(x^{1/2-\phi/4}), \\ a_2 &= d_1e_2 = (G - g)(H + h) = x^{1/2} + O(x^{1/2-\phi/4}), \\ b_2 &= d_2e_1 = (G + g)(H - h) = x^{1/2} + O(x^{1/2-\phi/4}), \end{aligned}$$

there is a  $(1/2 - \phi/4, C')$ -almost square  $n = a_1b_1 = a_2b_2$  of type 2 in the interval  $[x - C''x^{3/4-3\phi/8}, x + C''x^{3/4-3\phi/8}]$  for some  $C', C'' > 0$ . By picking  $\phi$  close to  $2/3$ , we have  $g(\theta) \leq 1/2$  for  $\theta > 1/3$ .

## 6 Integer almost divisible by some integer in an interval

Again let  $1/2 \leq \phi \leq 1$ . In the previous section, we found an interval of length  $x^{1-\phi}$  around  $x$  containing an integer divisible by some integer in the interval  $[x^{1-\phi}/2, x^{1-\phi}]$ . This is obviously true. Our goal in this section is to find a shorter interval still containing an integer divisible by some integer in the interval  $[x^{1-\phi}/2, x^{1-\phi}]$ . We hope that this will give some improvements to Theorem 3 (iv). Let us reformulate the question as follows:

**Question 1** *Let  $0 < \alpha \leq 1/2$  and  $X > 0$  be a large integer. Given  $0 < c_1 < c_2 \leq 1$ , find  $L$ , as small as possible, such that the interval  $[X - L, X]$  contains an integer that is divisible by some integer in the interval  $[c_1X^\alpha, c_2X^\alpha]$ .*

One may interpret the above as finding an integer in the interval  $[c_1 X^\alpha, c_2 X^\alpha]$  that almost divides  $X$  (with a remainder less than or equal to  $L$ ). We suspect the following

**Conjecture 2** *For any  $\epsilon > 0$ , one can take  $L = X^\epsilon$  in the above question as long as  $X$  is sufficiently large in terms of  $\epsilon$ .*

However, we can only prove

**Proposition 1** *Suppose  $(p, q)$  with  $0 \leq p \leq \frac{1}{2} \leq q \leq 1$  is an exponent pair for exponential sums. Then one can take  $L = X^{\frac{\alpha(q-p)}{1+p} + \frac{p}{1+p} + \epsilon}$  in the above question for any  $\epsilon > 0$  as long as  $X$  is sufficiently large in terms of  $\epsilon$ .*

Our method of proof is making use of Erdős-Turán inequality in the following form (see H.L. Montgomery [4, Corollary 1.2] for example):

**Lemma 1** *Suppose  $M$  is a positive integer chosen so that*

$$\sum_{l=1}^M \left| \sum_{j=1}^J e(lx_j) \right| \leq \frac{J}{10}.$$

*Then every arc  $\mathcal{J} = [\alpha, \beta] \subseteq [0, 1]$  of length  $\beta - \alpha \geq \frac{4}{M+1}$  contains at least  $\frac{1}{2}J(\beta - \alpha)$  of the points  $x_j$ ,  $1 \leq j \leq J$ . Here  $\|x\| = \min_{n \in \mathbb{Z}} |x - n|$ , the distance from  $x$  to the nearest integer, and  $e(x) = e^{2\pi i x}$ .*

Proof of Proposition 1: Our sequence  $\{x_j\}_{j=1}^J$  should be  $\{\frac{X}{a} : a \in \mathbb{Z} \text{ and } a \in [c_1 X^\alpha, c_2 X^\alpha]\}$ . We want to find some  $a$  such that the fractional part of  $\frac{X}{a}$  is small. For if  $\{\frac{X}{a}\} \in [0, \frac{4}{M}]$ , then  $\frac{X}{a} = k + \frac{\theta}{M}$  for some integer  $k$  and  $0 \leq \theta \leq 4$ . This gives  $X = ka + \frac{\theta}{M}a$  and  $X - \frac{\theta}{M}a = ka$ . Hence, with  $L = \frac{4c_2 X^\alpha}{M}$ , the interval  $[X - L, X]$  contains an integer that is divisible by some integer in  $[c_1 X^\alpha, c_2 X^\alpha]$ . Thus, in view of Lemma 1, it suffices to show

$$S := \sum_{l=K}^{2K} \left| \sum_{c_1 X^\alpha \leq a \leq c_2 X^\alpha} e\left(\frac{lX}{a}\right) \right| \leq X^{\alpha-\epsilon}$$

for any  $2K \leq M$  and  $\epsilon > 0$  as long as  $X$  is sufficiently large in terms of  $\epsilon$ . Keep in mind that we want  $M$  as large as possible.

By the theory of exponent pairs on exponential sums (see Chapter 3 section 4 of [4] for an overview),

$$\sum_{c_1 X^\alpha \leq a \leq c_2 X^\alpha} e\left(\frac{lX}{a}\right) \ll (lX(X^\alpha)^{-2})^p (X^\alpha)^q \ll K^p X^{p-2\alpha p + \alpha q} \quad (2)$$

if  $(p, q)$  with  $0 \leq p \leq \frac{1}{2} \leq q \leq 1$  is an exponent pair. Using (2), we have

$$\sum_{l=K}^{2K} \left| \sum_{c_1 X^\alpha \leq a \leq c_2 X^\alpha} e\left(\frac{lX}{a}\right) \right| \ll K^{1+p} X^{p-2\alpha p + \alpha q}.$$

Thus  $S \leq X^{\alpha-\epsilon}$  provided that, for  $X$  large enough,

$$K^{1+p} X^{p-2\alpha p+\alpha q} \leq X^{\alpha-\epsilon} \quad \text{or} \quad K \leq X^{\frac{\alpha(1-q+2p)}{1+p} - \frac{p}{1+p} - \epsilon}.$$

Therefore, we can pick

$$M = X^{\frac{\alpha(1-q+2p)}{1+p} - \frac{p}{1+p} - \epsilon} \quad \text{which gives} \quad L = 4c_2 X^{\frac{\alpha(q-p)}{1+p} + \frac{p}{1+p} + \epsilon}.$$

This proves Proposition 1 since  $\epsilon$  is arbitrary.

## 7 Proof of Theorem 3 (v)

Proof: We follow closely the proof of the third result. Applying Proposition 1 with  $\alpha = 1 - \phi$  and  $X = x + 3x^{\frac{(1-\phi)(q-p)}{1+p} + \frac{p}{1+p} + \epsilon}$ , the interval  $[x + x^{\frac{(1-\phi)(q-p)}{1+p} + \frac{p}{1+p} + \epsilon}, x + 3x^{\frac{(1-\phi)(q-p)}{1+p} + \frac{p}{1+p} + \epsilon}]$  contains an integer  $n = ab$  with integers  $a \in [x^{1-\phi}/2, x^{1-\phi}]$  and  $b \in [x^\phi, 3x^\phi]$ . Thus we can find

$$H \in [x^{\phi/2}, 3x^{\phi/2}] \quad \text{and} \quad G \in [x^{(1-\phi)/2}/2, x^{(1-\phi)/2}]$$

such that

$$0 < GH - \sqrt{x} \asymp x^{\frac{(1-\phi)(q-p)}{2(1+p)} + \frac{p}{2(1+p)} + \frac{\epsilon}{2}}.$$

Then the left hand side of (1),  $L = (GH - \sqrt{x})(GH + \sqrt{x}) \asymp x^{\frac{1+p+q}{2(1+p)} - \frac{q-p}{2(1+p)}\phi + \frac{\epsilon}{2}}$ .

Firstly we approximate  $L$  by  $g^2 H^2$ . For some choice of  $g \asymp x^{\frac{1+p+q}{4(1+p)} - \frac{2+p+q}{4(1+p)}\phi + \frac{\epsilon}{4}}$ , we have  $0 < L - g^2 H^2 \asymp g H^2 \asymp x^{\frac{1+p+q}{4(1+p)} + \frac{2+3p-q}{4(1+p)}\phi + \frac{\epsilon}{4}}$ . Note that we need  $\frac{1+p+q}{4(1+p)} - \frac{2+p+q}{4(1+p)}\phi \geq 0$  which means  $\phi \leq \frac{1+p+q}{2+p+q}$ .

Secondly we approximate  $L - g^2 H^2$  by  $h^2 G^2$ . For some choice of  $h \asymp x^{\frac{6+7p-q}{8(1+p)}\phi - \frac{3+3p-q}{8(1+p)} + \frac{\epsilon}{8}}$ , we have  $|L - g^2 H^2 - h^2 G^2| \ll h G^2 \asymp x^{\frac{5+5p+q}{8(1+p)} - \frac{2+p+q}{8(1+p)}\phi + \frac{\epsilon}{8}}$ . Note that  $\frac{6+7p-q}{8(1+p)}\phi - \frac{3+3p-q}{8(1+p)} \geq 0$  as  $\phi \geq 1/2$  and  $p, q \geq 0$ .

Thirdly, observe that  $g^2 h^2 \ll x^{\frac{3q-p-1}{4(1+p)} - \frac{3q-5p-2}{4(1+p)}\phi + \frac{3\epsilon}{4}} \ll x^{\frac{5+5p+q}{8(1+p)} - \frac{2+p+q}{8(1+p)}\phi + \frac{\epsilon}{8}}$  provided  $\phi < \frac{7+7p-5q}{6+11p-5q}$  and  $\epsilon$  is small enough. One can easily check that  $\frac{7+7p-5q}{6+11p-5q} > \frac{1+p+q}{2+p+q}$ . Therefore,  $|L - g^2 H^2 - h^2 G^2 + g^2 h^2| \ll_\epsilon x^{\frac{5+5p+q}{8(1+p)} - \frac{2+p+q}{8(1+p)}\phi + \frac{\epsilon}{8}}$  which gives

$$|x - d_1 d_2 e_1 e_2| = |x - (G - g)(G + g)(H - h)(H + h)| \ll_\epsilon x^{\frac{5+5p+q}{8(1+p)} - \frac{2+p+q}{8(1+p)}\phi + \frac{\epsilon}{8}}$$

provided  $\frac{1}{2} \leq \phi \leq \frac{1+p+q}{2+p+q}$ . Choose  $\phi = \frac{1+p+q}{2+p+q}$ , we have, after some simple algebra,

$$|x - d_1 d_2 e_1 e_2| = |x - (G - g)(G + g)(H - h)(H + h)| \ll_\epsilon x^{\frac{1}{2} + \frac{\epsilon}{8}}.$$



Now observe that with  $\phi = \frac{1+p+q}{2+p+q}$ , after some algebra,

$$GH - x^{1/2} \ll x^{\frac{q}{2(1+p)} - \frac{q-p}{2(1+p)}\phi + \frac{\epsilon}{2}} = x^{\frac{p+q}{2(2+p+q)} + \frac{\epsilon}{2}},$$

$$gH \ll x^{\frac{1+p+q}{4(1+p)} - \frac{2+p+q}{4(1+p)}\phi + \frac{\phi}{2} + \frac{\epsilon}{4}} = x^{\frac{1+p+q}{2(2+p+q)} + \frac{\epsilon}{4}},$$

$$hG \ll x^{\frac{6+7p-q}{8(1+p)}\phi - \frac{3+3p-q}{8(1+p)} + \frac{1-\phi}{2} + \frac{\epsilon}{8}} = x^{\frac{1+p+q}{2(2+p+q)} + \frac{\epsilon}{8}},$$

and

$$gh \ll x^{\frac{3q-p-1}{8(1+p)} - \frac{3q-5p-2}{8(1+p)}\phi + \frac{3\epsilon}{8}} = x^{\frac{p+q}{2(2+p+q)} + \frac{3\epsilon}{8}}.$$

Therefore  $a_1 = d_1e_1 = (G-g)(H-h)$ ,  $b_1 = d_2e_2 = (G+g)(H+h)$ ,  $a_2 = d_1e_2 = (G-g)(H+h)$  and  $b_2 = d_2e_1 = (G+g)(H-h)$  are all  $= x^{\frac{1}{2}} + O_\epsilon(x^{\frac{1+p+q}{2(2+p+q)} + \frac{\epsilon}{2}})$ . Therefore, there is a  $(\frac{1+p+q}{2(2+p+q)} + \frac{\epsilon}{2}, C_\epsilon)$ -almost square of type 2 in the interval  $[x - x^{\frac{1}{2} + \frac{\epsilon}{8}}, x + x^{\frac{1}{2} + \frac{\epsilon}{8}}]$ . This shows that  $g(\theta) \leq 1/2$  for  $\theta > \frac{1+p+q}{2(2+p+q)}$ . Since  $\frac{1+u}{2+u}$  is an increasing function, we try to find exponent pairs that make  $p+q$  as small as possible.

For example, recently Huxley [3] proved that  $(p, q) = (\frac{32}{205} + \epsilon, \frac{1}{2} + \frac{32}{205} + \epsilon)$  is an exponent pair for any  $\epsilon > 0$ . This gives  $\frac{1+p+q}{2(2+p+q)} \geq \frac{743}{2306} + \epsilon$  for any  $\epsilon > 0$  and hence Theorem 3 (v). Note that  $\frac{743}{2306} = 0.3222029488... < \frac{1}{3}$ . However, we still cannot beat the  $\frac{1}{2}$  bound for  $g(\theta)$ . Assuming the exponent pair conjecture that  $(\epsilon, \frac{1}{2} + \epsilon)$  is an exponent pair, we can push the range for  $\theta$  to  $\theta > 0.3$  with  $g(\theta) \leq \frac{1}{2}$  but this is still shy of the range  $\theta \geq \frac{1}{4}$ . Nevertheless, if one assumes Conjecture 2 in the previous section and imitates the proof of the (iv) or (v) of Theorem 3, one can get  $g(\theta) \leq \frac{1}{2}$  for  $\theta > \frac{1}{4}$ . This comes close to the conjecture  $g(1/4) = 1/2$ .

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